The triple category of categories

Christian Williams

University of California, Riverside

$\mathbb{C}\mathrm{at}$ has been understood as a double category: categories and functors, profunctors and transformations. Categories and profunctors are monads and bimodules in $\mathrm{Span}(\mathrm{Set})$: a profunctor from $\mathbb{X}$ to $\mathbb{A}$ is a span of sets ob$\mathbb{X} \leftarrow \mathcal{P} \rightarrow$ ob$\mathbb{A}$, giving for each pair of objects a set of "morphisms" $\mathcal{P}(\mathrm{X}, \mathrm{A})$, with a "precompose" action by $\mathbb{X}$ and a "postcompose" action by $\mathbb{A}$.

Yet there is also the richer notion of *two-sided fibration* [4]. The arrow category of $\mathbb{X}$ is a double category, i.e. a monad $\mathbb{X} \leftarrow \vec{\mathbb{X}} \rightarrow \mathbb{X}$ in $\mathrm{Span}(\mathrm{Cat})$, and a two-sided fibration is a bimodule of arrow categories: $\mathbb{X} \leftarrow \mathcal{R} \rightarrow \mathbb{Y}$ gives for each pair a category of morphisms and 2-morphisms $\mathcal{R}(\mathrm{x}, \mathrm{y})$, with precomposition by squares of $\vec{\mathbb{X}}$ and postcomposition by $\vec{\mathbb{Y}}$. Categories and two-sided fibrations form a tricategory [6].

We present a *triple category* of categories, in which the three kinds of 1-cell are functor, profunctor, and two-sided fibration. The three kinds of 2-cell are transformation, fibered functor, and *fibered profunctor* — a novel structure, connecting two-sided fibrations along profunctors, which we now define.

Just as arrow categories are monads in $\mathrm{Span}(\mathrm{Cat})$, a profunctor $\mathbb{X} \leftarrow \mathcal{P} \rightarrow \mathbb{A}$ gives an "arrow profunctor" which is a monad in $\mathrm{Span}(\mathrm{Prof})$: a span of profunctors, the apex consisting of commutative squares $\vec{\mathcal{P}}(\mathrm{x}, \mathrm{a}) = \{(p_0 : \mathcal{P}(\mathrm{X}_0, \mathrm{A}_0), p_1 : \mathcal{P}(\mathrm{X}_1, \mathrm{A}_1)) \mid \mathrm{a} \circ p_0 = p_1 \circ \mathrm{x}\}$; composition is that of squares.

$$\begin{tikzcd}
\mathbb{X} & \vec{\mathbb{X}} \arrow[l] \arrow[r] & \mathbb{X} \\
\mathcal{P} \arrow[u] \arrow[d] & \vec{\mathcal{P}} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \mathcal{P} \arrow[u] \arrow[d] \\
\mathbb{A} & \vec{\mathbb{A}} \arrow[l] \arrow[r] & \mathbb{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{X}_0 \arrow[rr, "\mathrm{x}"] \arrow[dd, "p_0"'] & & \mathrm{X}_1 \arrow[dd, "p_1"] \\
& p & \\
\mathrm{A}_0 \arrow[rr, "\mathrm{a}"'] & & \mathrm{A}_1
\end{tikzcd}$$

Definition. Let $\mathbb{X}, \mathbb{Y}, \mathbb{A}, \mathbb{B}$ be categories, let $\mathbb{X} \leftarrow \mathcal{P} \rightarrow \mathbb{A}$ and $\mathbb{Y} \leftarrow \mathcal{Q} \rightarrow \mathbb{B}$ be profunctors, and let $\mathbb{X} \leftarrow \mathcal{R} \rightarrow \mathbb{Y}$ and $\mathbb{A} \leftarrow \mathcal{S} \rightarrow \mathbb{B}$ be two-sided fibrations. A **fibered profunctor** over $\mathcal{P}, \mathcal{Q}$ from $\mathcal{R}$ to $\mathcal{S}$ is a profunctor $\mathcal{R} \leftarrow \mathcal{I} \rightarrow \mathcal{S}$ spanning from $\mathcal{P}$ to $\mathcal{Q}$, which is a $\vec{\mathcal{P}}, \vec{\mathcal{Q}}$-bimodule in $\mathrm{Span}(\mathrm{Prof})$.

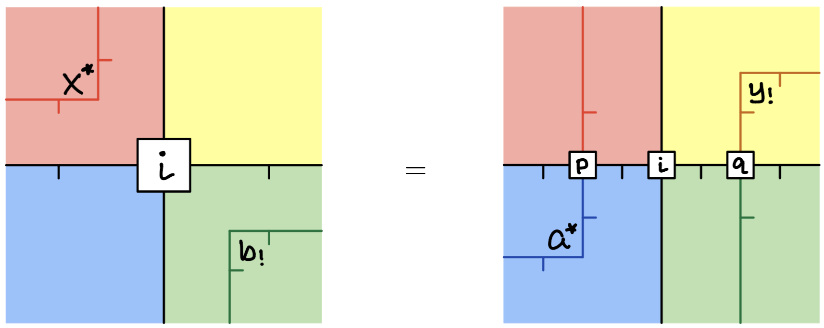
$$\begin{tikzcd}
\mathbb{X} & \mathcal{R} \arrow[l] \arrow[r] & \mathbb{Y} \\
\mathcal{P} \arrow[u] \arrow[d] & \mathcal{I} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \mathcal{Q} \arrow[u] \arrow[d] \\
\mathbb{A} & \mathcal{S} \arrow[l] \arrow[r] & \mathbb{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{X} \arrow[rr, "r"] \arrow[dd, "p"'] & & \mathrm{Y} \arrow[dd, "q"] \\
& \overset{\Vert}{i} \arrow[d, phantom, "\Downarrow"] & \\
\mathrm{A} \arrow[rr, "s"'] & & \mathrm{B}
\end{tikzcd}$$

Just as a fibration is a matrix of categories $\mathcal{R}(\mathrm{X}, \mathrm{Y})$, a fibered profunctor is a matrix of profunctors $\mathcal{I}(p, q)$: for each $p : \mathcal{P}(\mathrm{X}, \mathrm{A})$ and $q : \mathcal{Q}(\mathrm{Y}, \mathrm{B})$ there is a profunctor $\mathcal{R}(\mathrm{X}, \mathrm{Y}) \leftarrow \mathcal{I}(p, q) \rightarrow \mathcal{S}(\mathrm{A}, \mathrm{B})$ of "squares"; the actions of $\mathcal{R}$ and $\mathcal{S}$ give "sequential" composition, while the actions of $\mathcal{P}$ and $\mathcal{Q}$ give "parallel" composition.

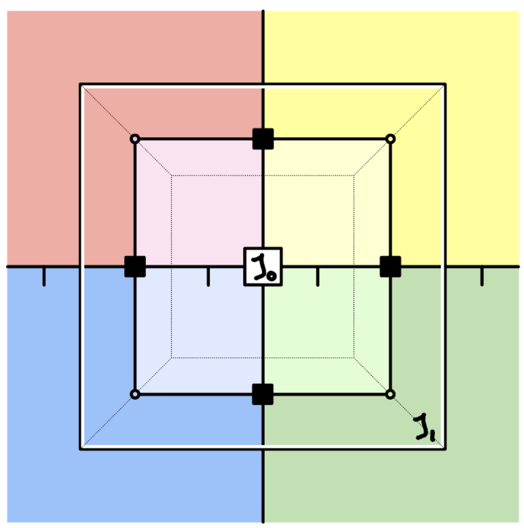
Note that the definition does not include any condition of coherence between the bimodule structure of $\mathcal{J}$ with that of $\mathcal{R}$ and $\mathcal{S}$. This is because the structures are uniquely determined: just as profunctor elements can be represented as morphisms via collage in $\mathbb{C}\mathrm{at}$ [4], fibered profunctors can be represented by squares and parallel composition, via collage in pseudomonads of $\mathbb{T}\mathrm{SFib}$: double categories with companions.

Theorem. Let $\mathcal{R} \leftarrow \mathcal{J} \rightarrow \mathcal{S}$ be the apex of a span of profunctors from $\mathcal{P}$ to $\mathcal{Q}$, as drawn above. If there exists a $\vec{\mathcal{P}}, \vec{\mathcal{Q}}$-bimodule structure of $\mathcal{J}$, it is uniquely isomorphic to the fibered profunctor $\mathcal{P} \leftarrow [\mathcal{J}] \rightarrow \mathcal{Q}$ in which elements are squares of a double category, and action is composition by squares of $\vec{\mathcal{P}}$ and $\vec{\mathcal{Q}}$.

So a fibered profunctor is canonically natural with respect to its source and target two-sided fibrations.



So the naïve view of profunctor elements as morphisms expands to two dimensions: two-sided fibrations give horizontal morphisms, and fibered profunctors give squares. This makes a powerful visual language: fibered profunctors can be drawn as beads, and their transformations can be drawn as *beads within beads*.



A fibered transformation $\mathcal{J}_0 \rightarrow \mathcal{J}_1$.

We extend the language of string diagrams [2] to triple categories. Above, the back and front face are fibered profunctors, left and right are transformations, top and bottom are fibered functors, and the cube is a **fibered transformation**: a bimodule transformation $\mathcal{J}_0 \rightarrow \mathcal{J}_1$ in $\mathrm{Span}(\mathrm{Prof})$.

Theorem. Categories, functors, profunctors, and two-sided fibrations form a triple category $\mathbb{C}\mathrm{at}_\Omega$.

Composition of two-sided fibrations is associative up to isomorphism, and unital up to equivalence.

Using string diagrams as a template, the language of $\mathbb{C}\mathrm{at}_\Omega$ is three-dimensional, combining the co/end calculus of profunctors [3] and the "co/descent calculus" of two-sided fibrations [1]. It is a rich universe, in which category theory and fibered category theory are unified.

References

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